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NEW HIGH ORDER THEORY FOR FUNCTIONALLY GRADED SHELLS

New theory for functionally graded (FG) shell based on expansion of the equations of elasticity for functionally graded materials (GFMs) into Legendre polynomials series has been developed. Stress and strain tensors, vectors of displacements, traction and body forces have been expanded into Legendre polynomials series in a thickness coordinate. In the same way functions that describe functionally graded relations has been also expanded. Thereby all equations of elasticity including Hook's law have been transformed to corresponding equations for Fourier coefficients. Then system of differential equations in term of displacements and boundary conditions for Fourier coefficients has been obtained. Cases of the first and second approximations have been considered in more details. For obtained boundary-value problems solution finite element (FE) has been used. Numerical calculations have been done with Comsol Multiphysics and Matlab.

Keywords: shell, FEM, FGM, Legendre polynomial.

The FG thin-walled structures have numerous applications, especially in reactor vessels, turbines and many other applications in aerospace engineering [4–6]. Laminated composite materials are commonly used in many kinds of engineering structures. In conventional laminated composite structures, homogeneous elastic laminas are bonded together to obtain enhanced mechanical properties. However, the abrupt change in material properties across the interface between different materials can result in large interlaminar stresses leading to delamination. One way to overcome these adverse effects is to use FGMs in which material properties vary continuously by gradually changing the volume fraction of the constituent materials. This eliminates interface problems of composite materials and thus the stress distributions are smooth.

In this paper we are developing new theory for FG shells based on expansion of the equations of elasticity for GFMs into Legendre polynomials series[1, 2, 7]. We explored such an approach in our previous publications for solution thermoelastic contact problems [8–12]. More specifically, here we expanded functions that describe functionally graded relations into Legendre polynomials series and find Hooke's law that related Fourier coefficients for expansions of stress and strain Numerical examples are presented.

1. 3-D formulation. Let a linear elastic body occupy an open in 3-D Euclidian space simply connected bounded domain $V \in \mathbf{R}^3$ with a smooth boundary ∂V. We assume that elastic body is inhomogeneous isotropic shell of arbitrary geometry with $2h$ thickness. The domain is $V = \Omega \times [-h, h]$ and it is embedded in in Euclidean space. Boundary of the shell can be presented in the form $\partial V = S \cup \Omega^+ \cup \Omega^-$. Here Ω is the middle surface of the shell, $\partial\Omega$ is its boundary, Ω^+ and Ω^{--} are the outer sides and $S = \Omega \times [-h, h]$ is a sheer side.

Stress-strain state of the elastic body is defined by stress σ^{ij} and ε_{ij} strain tensors and displacements u_i, traction p_i, and body forces b_i vectors. These quantities are not independent, they are related by equations of elasticity.

For convenience we transform above equations of elasticity taking into account that the radius vector $\mathbf{R}(\mathbf{x})$ of any point in domain *V*, occupied by material points of shell may be presented as

$$\mathbf{R}(\mathbf{x}) = \mathbf{r}(\mathbf{x}_\alpha) + x_3 \mathbf{n}(\mathbf{x}_\alpha) \qquad (1.1)$$

where $\mathbf{r}(\mathbf{x}_\alpha)$ is the radius vector of the points located on the middle surface of shell, $\mathbf{n}(\mathbf{x}_\alpha)$ is a unit vector normal to the middle surface.

Let us consider that $\mathbf{x}_\alpha = (x^1, x^2)$ are curvilinear coordinates associated with main curvatures of the middle surface of the shell. In order to simplify 3-D equations of elasticity we introduce orthogonal system of coordinates related to main curvatures of the middle surface of the shell. Such coordinates are widely used in the shell theory. In this case the equations of equilibrium have the form

$$\frac{\partial(A_2\sigma_{11})}{\partial x_1} + \frac{\partial(A_1\sigma_{12})}{\partial x_2} + A_1A_2\frac{\partial\sigma_{13}}{\partial x_3} + \sigma_{12}\frac{\partial A_1}{\partial x_2} + \sigma_{13}A_1A_2k_1 - \sigma_{22}\frac{\partial A_2}{\partial x_1} + A_1A_2b_1 = 0,$$

$$\frac{\partial(A_2\sigma_{21})}{\partial x_1} + \frac{\partial(A_1\sigma_{22})}{\partial x_2} + A_1A_2\frac{\partial\sigma_{23}}{\partial x_3} + \sigma_{21}\frac{\partial A_2}{\partial x_1} + \sigma_{23}A_1A_2k_2 - \sigma_{11}\frac{\partial A_1}{\partial x_2} + A_1A_2b_2 = 0,$$

$$\frac{\partial(A_2\sigma_{31})}{\partial x_1} + \frac{\partial(A_1\sigma_{32})}{\partial x_2} + \frac{\partial(A_1A_2\sigma_{33})}{\partial x_3} - \sigma_{11}A_1A_2k_1 - \sigma_{22}A_1A_2k_2 + A_1A_2b_3 = 0. \qquad (1.2)$$

Cauchy relations have the form

$$\varepsilon_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}u_2 + k_1u_3, \qquad \varepsilon_{22} = \frac{1}{A_2}\frac{\partial u_2}{\partial x_2} + \frac{1}{A_2A_1}\frac{\partial A_2}{\partial x_1}u_1 + k_2u_3,$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3}, \qquad \varepsilon_{12} = \frac{1}{A_2}\left(\frac{\partial u_1}{\partial x_2} - \frac{1}{A_1}\frac{\partial A_2}{\partial x_1}u_2\right) + \frac{1}{A_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{1}{A_2}\frac{\partial A_1}{\partial x_2}u_1\right),$$

$$\varepsilon_{13} = \frac{\partial u_1}{\partial x_3} - k_1 u_1 + \frac{1}{A_1}\frac{\partial u_3}{\partial x_1}, \qquad \varepsilon_{23} = \frac{\partial u_2}{\partial x_3} - k_2 u_2 + \frac{1}{A_2}\frac{\partial u_3}{\partial x_2} \qquad (1.3)$$

Here $A_\alpha(x_1, x_2) = \sqrt{\mathbf{r}(x_1, x_2)\mathbf{r}(x_1, x_2)}$ are coefficients of the first quadratic form of the middle surface of the shell, $k_\alpha(x_1, x_2)$ are it main curvatures.

In the case if inhomogeneous of the shell consists of graduation of the elastic modulus in the x_3 direction generalized Hook's law for FG elastic shell we represent in theform

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}), \qquad c_{ijkl}(x) = E(x)\, c^0_{ijkl}, \qquad (1.4)$$

where for isotropic shell

$$c^0_{ijkl} = \lambda^0\delta_{ij}\delta_{kl} + 2\mu^0\delta_{il}\delta_{jk}, \qquad \mu^0 = 1\big/\big(2\,(1+\nu)\big), \qquad \lambda^0 = 2\nu\mu^0\big/(1-2\nu). \quad (1.5)$$

Substituting Cauchy relations (1.3) in Hook's law (1.4) and then Hook's law into equations of equilibrium (1.2) we obtain differential equations of equilibrium in the form of displacements

$$A_{ij}(\boldsymbol{x})u_j(\boldsymbol{x}) + b_i(\boldsymbol{x}) = 0\,. \qquad (1.6)$$

Here

$$A_{ij}(x) = E(x)c^0_{ijkl}\partial_k\partial_l = E(x)A^0_{ij}, \qquad (1.7)$$

where A^0_{ij} is a differential operator that correspond to the case of homogeneous equations of elasticity. These equations will be used for elaboration of the 2-D equations for FG shells.

2. 2-D formulation. Let us expand the parameters, that describe stress-strain of the cylindrical shell in the Legendre polynomials series along the coordinate x_3.

$$u_i(x) = \sum_{k=0}^{\infty} u_i^k(\boldsymbol{x}_\alpha)P_k(\omega), \qquad u_i^k(\boldsymbol{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} u_i(\boldsymbol{x}_\alpha, x_3)P_k(\omega)dx_3,$$

$$\sigma_{ij}(x) = \sum_{k=0}^{\infty} \sigma_{ij}^k(\boldsymbol{x}_\alpha)P_k(\omega), \qquad \sigma_{ij}^k(\boldsymbol{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} \sigma_{ij}(\boldsymbol{x}_\alpha, x_3)P_k(\omega)dx_3,$$

$$\varepsilon_{ij}(x) = \sum_{k=0}^{\infty} \varepsilon_{ij}^k(\boldsymbol{x}_\alpha)P_k(\omega), \qquad \varepsilon_{ij}^k(\boldsymbol{x}_\alpha) = \frac{2k+1}{2h}\int_{-h}^{h} \varepsilon_{ij}(\boldsymbol{x}_\alpha, x_3)P_k(\omega)dx_3,$$

$$p_i(x) = \sum_{k=0}^{\infty} p_i^k\left(\boldsymbol{x}_\alpha\right) P_k(\omega), \qquad p_i^k\left(\boldsymbol{x}_\alpha\right) = \frac{2k+1}{2h}\int_{-h}^{h} p_i\left(\boldsymbol{x}_\alpha, x_3\right) P_k(\omega) dx_3,$$

$$b_i(x) = \sum_{k=0}^{\infty} b_i^k\left(\boldsymbol{x}_\alpha\right) P_k(\omega), \qquad b_i^k\left(\boldsymbol{x}_\alpha\right) = \frac{2k+1}{2h}\int_{-h}^{h} b_i\left(\boldsymbol{x}_\alpha, x_3\right) P_k(\omega) dx_3. \quad (2.1)$$

Substituting these expansions in equations (1.2) – (1.3) we obtain corresponding relations for Legendre polynomials series coefficients. Equations of equilibrium have the form

$$\frac{\partial\left(A_2\sigma_{11}^k\right)}{\partial x_1} + \frac{\partial\left(A_1\sigma_{12}^k\right)}{\partial x_2} + \sigma_{12}^k\frac{\partial A_1}{\partial x_2} + \sigma_{13}^k A_1 A_2 k_1 - \sigma_{22}^k\frac{\partial A_2}{\partial x_1} - \underline{\sigma_{13}^k} + A_1 A_2 f_1^k = 0,$$

$$\frac{\partial\left(A_2\sigma_{21}^k\right)}{\partial x_1} + \frac{\partial\left(A_1\sigma_{22}^k\right)}{\partial x_2} + \sigma_{12}^k\frac{\partial A_2}{\partial x_1} + \sigma_{23}^k A_1 A_2 k_2 - \sigma_{11}^k\frac{\partial A_2}{\partial x_1} - \underline{\sigma_{23}^k} + A_1 A_2 f_2^k = 0,$$

$$\frac{\partial\left(A_2\sigma_{31}^k\right)}{\partial x_1} + \frac{\partial\left(A_1\sigma_{32}^k\right)}{\partial x_2} - \sigma_{11}^k A_1 A_2 k_1 - \sigma_{22}^k A_1 A_2 k_2 - \underline{\sigma_{33}^k} + A_1 A_2 f_3^k = 0, \quad (2.2)$$

where

$$\underline{\sigma_{i3}^k}\left(\boldsymbol{x}_\alpha\right) = A_1 A_2 \frac{2k+1}{h}\left(\sigma_{i3}^{k-1}\left(\boldsymbol{x}_\alpha\right) + \sigma_{i3}^{k-3}\left(\boldsymbol{x}_\alpha\right) + \ldots\right),$$

$$f_i^k\left(\boldsymbol{x}_\alpha\right) = b_i^k\left(\boldsymbol{x}_\alpha\right) + \frac{2k+1}{h}\left(\sigma_{i3}^{+}\left(\boldsymbol{x}_\alpha\right) - (-1)^k\sigma_{i3}^{-}\left(\boldsymbol{x}_\alpha\right)\right). \quad (2.3)$$

Cauchy relations have the form

$$\varepsilon_{11}^k = \frac{1}{A_1}\frac{\partial u_1^k}{\partial x_1} + \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial x_2}u_2^k + k_1 u_3^k,$$

$$\varepsilon_{22}^k = \frac{1}{A_2}\frac{\partial u_2^k}{\partial x_2} + \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial x_1}u_1^k + k_2 u_3^k,$$

$$\varepsilon_{12}^k = \frac{1}{A_2}\left(\frac{\partial u_1^k}{\partial x_2} - \frac{1}{A_1}\frac{\partial A_2}{\partial x_1}u_2^k\right) + \frac{1}{A_1}\left(\frac{\partial u_2^k}{\partial x_1} - \frac{1}{A_2}\frac{\partial A_1}{\partial x_2}u_1^k\right),$$

$$\varepsilon_{13}^k = \frac{1}{A_1}\frac{\partial u_3^k}{\partial x_1} - k_1 u_1^k + \underline{u_1^k}, \qquad \varepsilon_{13}^k = \frac{1}{A_1}\frac{\partial u_3^k}{\partial x_1} - k_1 u_1^k + \underline{u_1^k}, \qquad \varepsilon_{33} = \underline{u_3^k}, \quad (2.4)$$

where

$$\underline{u_i^k}\left(\boldsymbol{x}_\alpha\right) = \frac{2k+1}{h}\left(u_i^{k+1}\left(\boldsymbol{x}_\alpha\right) + u_i^{k+3}\left(\boldsymbol{x}_\alpha\right) + \ldots\right). \quad (2.5)$$

In order to transform Hook's law in 1-D form we expand Young's $E(\mathbf{x})$ in Legendre polynomials series

$$E(x) = \sum_{r=1}^{\infty} E^r\left(\boldsymbol{x}_\alpha\right) P_r(x_3)\,, \qquad E^k\left(\boldsymbol{x}_\alpha\right) = \frac{2k+1}{2h}\int_{-h}^{h} E\left(\boldsymbol{x}_\alpha,\ x_3\right) P_k(\omega) dx_3\,. \tag{2.6}$$

Substituting this expansion and expansions for stress and strain tensors in Hook's law we obtain 1-D Hook's law for Legendre polynomials series coefficients

$$\sigma_{ij}^n\left(\boldsymbol{x}_\alpha\right) = c_{ijkl}^0 \sum_{r=1}^{\infty}\sum_{m}^{\infty} \in^{nrm}\ E^r\left(\boldsymbol{x}_\alpha\right) \varepsilon_{kl}^m\left(\boldsymbol{x}_\alpha\right), \tag{2.7}$$

where

$$\in^{nrm} = \int_{-1}^{1} P_n\left(x_3\right) P_r\left(x_3\right) P_m\left(x_3\right) dx_3\,. \tag{2.8}$$

Substituting Cauchy relations (2.4) and Hook's law (2.7) in equations of equilibrium (2.2) we obtain differential equations in displacements. This system of equations contains infinite number of equations which are 2-D, they can be written in the form

$$\frac{\partial}{\partial x_1}\left[\tilde{C}_{ij}\left(x_1\right)\frac{\partial u_j^k\left(x_1\right)}{\partial x_1}\right] + \tilde{B}_{ij}\left(x_1\right)\frac{\partial u_j^k\left(x_1\right)}{\partial x_1} + \tilde{A}_{ij}\left(x_1\right) u_j^k\left(x_1\right) = f_i^k\left(x_1\right). \tag{2.9}$$

Here infinite dimensional matrixes have the form $\tilde{\mathbf{C}} = \mathbf{E}\cdot\mathbf{C}$, $\tilde{\mathbf{B}} = \mathbf{E}\cdot\mathbf{B}$ and $\tilde{\mathbf{A}} = \mathbf{E}\cdot\mathbf{A}$;

$$\mathbf{C} = \begin{vmatrix} C_{ij} & 0 & \cdots \\ 0 & C_{ij} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}, \quad \mathbf{B} = \begin{vmatrix} B_{ij}^{00} & B_{ij}^{01} & \cdots \\ B_{ij}^{10} & B_{ij}^{11} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}, \quad \mathbf{A} = \begin{vmatrix} A_{ij}^{00} & A_{ij}^{01} & \cdots \\ A_{ij}^{10} & A_{ij}^{11} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}, \quad \mathbf{u} = \begin{vmatrix} u_j^0 \\ u_j^1 \\ \vdots \end{vmatrix}, \quad \mathbf{f} = \begin{vmatrix} f_j^0 \\ f_j^1 \\ \vdots \end{vmatrix} \tag{2.10}$$

Matrixes **C**, **B** and **A** correspond to the case of homogeneous elastic shells, matrix **E** characterized inhomogeneous properties of the shell.

$$\mathbf{E} = \begin{vmatrix} E_{ij}^{00} & E_{ij}^{01} & \cdots \\ E_{ij}^{10} & E_{ij}^{11} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix}, \qquad E_{ij}^{nm} = \begin{vmatrix} E^{nm} & 0 \\ 0 & E^{nm} \end{vmatrix} \tag{2.11}$$

where $E^{nm} = \in^{nrm} E^r$. Now instead of one 3-D system of the differential equations in displacements (1.6) we have of 2-D infinite differential equations for coefficients of the Legandre's polynomial series expansion. In order to simplify the problem approximate theory has to be developed and only finite set of members have to be taken into account in the expansion (2.1). Order of the system of equations depends on assumption regarding thickness distribution of the stress-strain parameters of the shell.

3. Results and discussion. We consider here the case of relatively thick axisymmetric cylindrical shells. Therefore we will keep three members in polynomial expansion (2.1). In this case we will get the second order approximation equations for functionally graded shells. In this case the stress-strain parameters, which describe the state of the shell, can be presented in the form

$$\sigma_{ij}(x)=\sigma_{ij}^{0}\left(\boldsymbol{x}_{\alpha}\right)P_0(\omega)+\sigma_{ij}^{1}\left(\boldsymbol{x}_{\alpha}\right)P_1(\omega)+\sigma_{ij}^{2}\left(\boldsymbol{x}_{\alpha}\right)P_2(\omega)\ ,$$

$$\varepsilon_{ij}(x)=\varepsilon_{ij}^{0}\left(\boldsymbol{x}_{\alpha}\right)P_0(\omega)+\varepsilon_{ij}^{1}\left(\boldsymbol{x}_{\alpha}\right)P_1(\omega)+\varepsilon_{ij}^{2}\left(\boldsymbol{x}_{\alpha}\right)P_2(\omega)\ ,$$

$$u_i(x)=u_i^{0}\left(\boldsymbol{x}_{\alpha}\right)P_0(\omega)+u_i^{1}\left(\boldsymbol{x}_{\alpha}\right)P_1(\omega)+u_i^{2}\left(\boldsymbol{x}_{\alpha}\right)P_2(\omega)\ ,$$

$$p_i(x)=p_i^{0}\left(\boldsymbol{x}_{\alpha}\right)P_0(\omega)+p_i^{1}\left(\boldsymbol{x}_{\alpha}\right)P_1(\omega)+p_i^{2}\left(\boldsymbol{x}_{\alpha}\right)P_2(\omega)\ ,$$

$$b_i(x)=b_i^{0}\left(\boldsymbol{x}_{\alpha}\right)P_0(\omega)+b_i^{1}(\omega)P_1(\omega)+b_i^{2}\left(\boldsymbol{x}_{\alpha}\right)P_2(\omega)\ . \qquad (3.1)$$

Taking into account formulae (2.8) for the coefficients $\in^{nrm}$ Hook's law for coefficients of the Legendre polynomials series expansion(2.7) has the form

$$\sigma_{ij}^{0}=c_{ijkl}^{0}\left(2E^{0}\varepsilon_{kl}^{0}+\frac{2}{3}E^{1}\varepsilon_{kl}^{1}+\frac{2}{5}E^{2}\varepsilon_{kl}^{2}\right),$$

$$\sigma_{ij}^{1}=c_{ijkl}^{0}\left(\frac{2}{3}E^{1}\varepsilon_{kl}^{0}+\left(\frac{2}{3}E^{0}+\frac{4}{15}E^{2}\right)\varepsilon_{kl}^{1}+\frac{4}{15}E^{1}\varepsilon_{kl}^{2}\right),$$

$$\sigma_{ij}^{2}=c_{ijkl}^{0}\left(\frac{2}{5}E^{2}\varepsilon_{kl}^{0}+\frac{4}{15}E^{1}\varepsilon_{kl}^{1}+\left(\frac{2}{5}E^{0}+\frac{4}{35}E^{2}\right)\varepsilon_{kl}^{2}\right). \qquad (3.2)$$

Now system of equations for displacements has the same form as (2.9), but it contains only four equations and corresponding matrixes and vector have the form

$$
\mathbf{E}=\begin{vmatrix}
2E^0 & 0 & \frac{2}{3}E^1 & 0 & \frac{2}{5}E^2 & 0\\
0 & 2E^0 & 0 & \frac{2}{3}E^1 & 0 & \frac{2}{5}E^2\\
\frac{2}{3}E^1 & 0 & \frac{2}{3}E^0+\frac{4}{15}E^2 & 0 & \frac{4}{15}E^1 & 0\\
0 & \frac{2}{3}E^1 & 0 & \frac{2}{3}E^0+\frac{4}{15}E^2 & 0 & \frac{4}{15}E^1\\
\frac{2}{5}E^2 & 0 & \frac{4}{15}E^1 & 0 & \frac{2}{5}E^0+\frac{4}{35}E^2 & 0\\
0 & \frac{2}{5}E^2 & 0 & \frac{4}{15}E^1 & 0 & \frac{2}{5}E^0+\frac{4}{35}E^2
\end{vmatrix}, \quad (3.3)
$$

$$
\mathbf{B}=\begin{vmatrix}
0 & \frac{\lambda}{R} & 0 & \frac{\lambda}{h} & 0 & 0\\
-\frac{\lambda}{R} & 0 & \frac{\mu}{h} & 0 & 0 & 0\\
0 & -\frac{3\mu}{h} & 0 & \frac{\lambda}{R} & 0 & \frac{3\lambda}{h}\\
-\frac{3\lambda}{h} & 0 & -\frac{\lambda}{R} & 0 & \frac{3\mu}{h} & 0\\
0 & 0 & 0 & -\frac{5\mu}{h} & 0 & \frac{\lambda}{R}\\
0 & 0 & -\frac{5\lambda}{h} & 0 & -\frac{\lambda}{R} & 0
\end{vmatrix}, \quad
\mathbf{u}=\begin{vmatrix} u_1^0\\ u_3^0\\ u_1^1\\ u_3^1\\ u_1^2\\ u_3^2 \end{vmatrix}, \quad
\mathbf{f}=\begin{vmatrix} f_1^0\\ f_3^0\\ f_1^1\\ f_3^1\\ f_1^2\\ f_3^2 \end{vmatrix} \quad (3.4)
$$

$$
\mathbf{A}=\begin{vmatrix}
0 & 0 & 0 & 0 & 0 & 0\\
0 & -\frac{\lambda+2\mu}{R^2} & 0 & -\frac{\lambda}{Rh} & 0 & 0\\
0 & 0 & -\frac{3\mu}{h^2} & 0 & 0 & 0\\
0 & -\frac{3\lambda}{Rh} & 0 & -(\lambda+2\mu)\left(\frac{1}{R^2}+\frac{3}{h^2}\right) & 0 & -\frac{3\lambda}{Rh}\\
0 & 0 & 0 & 0 & -\frac{15\mu}{h^2} & 0\\
0 & 0 & -\frac{5\lambda}{Rh} & 0 & 0 & -(\lambda+2\mu)\left(\frac{1}{R^2}+\frac{15}{h^2}\right)
\end{vmatrix} \quad (3.5)
$$

$$\mathbf{C} = \begin{vmatrix} \lambda+2\mu & 0 & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda+2\mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{vmatrix} . \tag{3.6}$$

Substituting these matrices into (2.9) we obtain system of differential equations which together with corresponding boundary conditions can be used for the stress-strain calculation for the second approximation shell theory.

Material properties of an FGM are the functions of volume fractions and they are managed by a volume fraction. When the shell is considered to consist of two materials with Young's modulus E_1 and E_2 respectively, the effective Young's modulus $E(x_3)$ given by the following power-law expression

$$E(x_3) = \left(E_2 - E_1\right)\left(\frac{x_3 + h}{2h}\right)^n + E_1 \quad (n \geq 0) . \tag{3.7}$$

Substituting function (3.7) into equation (2.6) we obtain expressions for the Legendre polynomials coefficients for the effective Young's modulus

$$E^1 = \frac{\left(E_2 + E_1\right)n}{1+n}, \qquad E^2 = \frac{\left(E_2 - E_1\right)nh}{2+3n+n^2}, \qquad E^3 = -\frac{5\left(E_2 - E_1\right)(n-1)nh^2}{(1+n)(2+n)(3+n)} . \tag{3.8}$$

For simplicity in this study we consider dimensionless coordinates $\xi_1 = x_1/L$ and $\xi_3 = x_3/h$ have been introduced. Calculations have been done for Young's modulus equal to $E_1 = 1\ Pa$ and $E_1/E_2 = 2$ and for Poison ratio $\nu = 0.3$ respectively, other parameters are $R = 0.25\,L$, $h = 0.25\,R$ and $n = 0.2$. Numerical calculations have been done using commercial software Comsol Multiphysics and Matlab. Results of calculations are presented on Fig. 1–3.

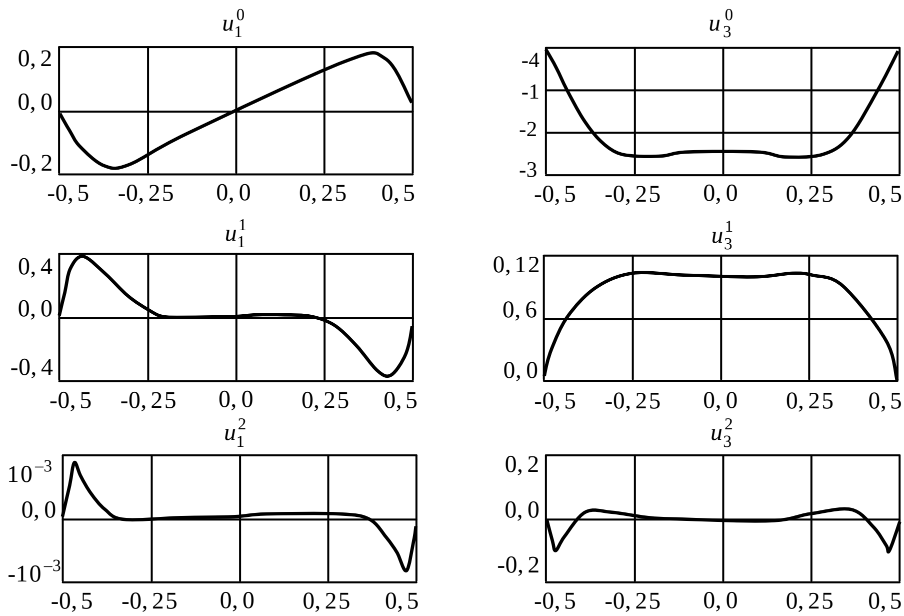

Fig.1

Fig. 1 shows the Legendre polynomials coefficients for the displacements distribution versus the normalized length for the second approximation theory. These coefficients are FEM solutions of the systems of differential equations (2.9)

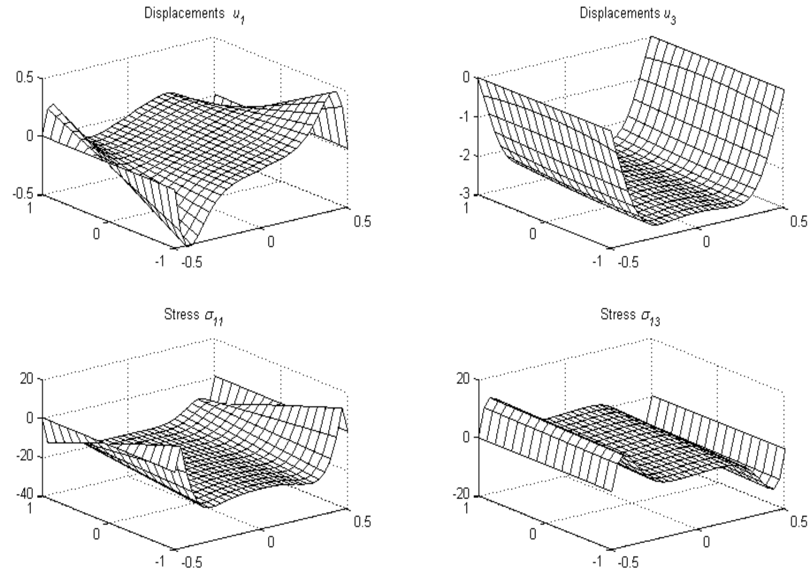

Fig.2

with matrix operators (3.3)–(3.4). Fig. 2 shows displacements and stresses distribution versus normalized length and thickness for second approximation theory.

РЕЗЮМЕ. Разработана новая теория для функционально градиентных (неоднородных по толщине) оболочек основанная на разложении трехмерных уравнений теории упругости для неднородного тела в ряды по полиномах Лежандра. Более подробно рассмотрены случаи первого и второго приближений. Четко сформулированы все необходимые уравнения для коэффициентов Фурье указанного разложения, а также соответствующие краевые задачи. Для численного решения сформулированной проблемы применялся метод конечных элементов (МКЭ) и использовалось коммерческое программное обеспечение COMSOL Multiphysics и Matlab. Для проверки предложенной теории и полученных уравнений производилось сравнение с результатами, полученными с помощью уравнений теории упругости с использованием экспоненциального закона изменения свойств материала по толщине для случая осесимметричной цилиндрической оболочки. Изучено влияние различных параметров на напряженно-деформированное состояние осесимметричной цилиндрической оболочки.

Ключевые слова: оболочка, МНК , МКЭ, полиномы Лежандра.

РЕЗЮМЕ. Розроблено нову теорію для осесиметричної циліндричної оболонки на основі розширення осесиметричних рівнянь теорії пружності для МНК в поліномах Лежандра. Більш докладно розглянуті випадки першого і другого наближень. Чітко сформульовані всі необхідні рівняння і спадкові коефіцієнти, а також відповідні крайові задачі. Для чисельного рішення сформульованої проблеми застосовувався метод кінцевих елементів (МКЕ) з використанням комерційного програмного забезпечення COMSOL Multiphysics і Matlab. Для перевірки запропонованої теорії та отриманих рівнянь для порівняння з результатами, отриманими за допомогою рівнянь теорії пружності, було зроблено виключення із закону експоненційної функції. Вивчено вплив різних параметрів на напружено-деформований стан циліндричної оболонки.

Ключові слова: оболонка, МНК, МСЕ, поліноми Лежандра.

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